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2007 J. Phys. A: Math. Theor. 40 6329

(http://iopscience.iop.org/1751-8121/40/24/005)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 6329–6348 doi:10.1088/1751-8113/40/24/005

Phase portraits of cubic polynomial vector fields of Lotka–Volterra type having a rational first integral of degree 2

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Received 9 January 2007, in final form 5 April 2007
Published 30 May 2007
Online at stacks.iop.org/JPhysA/40/6329

Abstract

We classify all the global phase portraits of the cubic polynomial vector fields of Lotka–Volterra type having a rational first integral of degree 2. For such vector fields there are exactly 28 different global phase portraits in the Poincaré disc up to a reversal of sense of all orbits.

PACS number: 02.30.Hq
Mathematics Subject Classification: 34C05, 34A34, 34C14

1. Introduction

Let P and Q be two real polynomials in the variables x and y, then we say that $X = (P, Q) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a *cubic polynomial vector field* if the maximum of the degrees of the polynomials P and Q is 3. We say that a cubic polynomial vector field is of *Lotka–Volterra type* if x is a factor of P and y is a factor of Q.

Lotka–Volterra systems typically model the time evolution of conflicting species in biology [20]. They have been largely studied starting with Lotka [18] and Volterra [23]. There are many other natural phenomena modelled by Lotka–Volterra systems, such as the coupling of waves in laser physics [15], the evolution of electrons, ions and neutral species in plasma physics [16]. In hydrodynamics they model the convective instability in the Benard problem [8]. Similarly, they appear in the interaction of gases in a background host medium [19]. In the theory of partial differential equations they can be obtained as a discretized form of the Korteweg–de Vries equation [5]. They also play a role in such diverse topics of current interest as neural networks [21], biochemical reactions, etc. Their interest becomes crucial after the work of Brenig and Goriely [6, 7], where they prove that a large class of ordinary differential equations implied in various fields of physics, biology, chemistry and economics can be transformed into a three-dimensional Lotka–Volterra system using a quasimonomial formalism. In the context of plasma physics, all the nonlinear terms represent binary interactions or model certain transport across the boundary of the system. Typically in all these applications, the Lotka–Volterra systems are taken quadratic. Further models [13] were introduced, generalizing the Lotka–Volterra systems, to model the interaction among biochemical populations. Cubic polynomial Lotka–Volterra vector fields have appeared explicitly modelling certain phenomena arising in oscillating chemical reactions as the so-called Lotka–Volterra–Brusselator (see [12]) and in well-known predator–prey models that give rise to periodic variations in the populations (see [11, 17, 22]), etc.

The main goal of this paper is to classify the global phase portraits in the Poincaré disc (see subsection 3.2) of the cubic polynomial vector fields (P, Q) of Lotka–Volterra type having a rational first integral H of degree 2 and with P and Q coprime.

If $f = f(x, y)$ and $g = g(x, y)$ are polynomials then we say that the function f/g is *rational*. If the maximum of the degrees of f and g is 2 and f and g are coprime, then we say that f/g is a rational function of *degree* 2.

In this paper we do not allow that vector fields (P, Q) be quadratic, because the quadratic systems having such kind of first integrals have been studied recently in [10]. Moreover as in that paper we do not allow that H or $1/H$ be a polynomial because the quadratic Lotka–Voltera systems having a polynomial first integral have been studied in [9].

We note that the cubic polynomial vector fields having a rational first integral of degree 2 have all their orbits contained in conics. Therefore, their orbits are very simple curves but this does not prevent that their phase portraits can present a rich variety of dynamics as is illustrated in our main result.

Theorem 1. *The phase portrait of a planar cubic polynomial vector field of Lotka–Volterra type with a rational first integral of degree 2 and with P and Q coprime, or the phase portrait with the sense of all orbits reversed, is topologically equivalent to one of the 28 phase portraits described in figure 1.*

The paper is organized as follows. In section 2 we characterize the cubic polynomial vector fields of Lotka–Volterra type having a rational first integral of degree 2. In section 3 we present the basic results on singular points, Poincaré compactification and homogeneous quadratic vector fields that we shall need. The rest of the sections are dedicated to the search of these phase portraits.

2. Characterization of cubic polynomial vector fields of Lotka–Volterra type

The cubic polynomial vector fields of Lotka–Volterra type having a rational first integral of degree 2 which is not polynomial are characterized in the next result.

Proposition 2. *A Lotka–Volterra cubic polynomial vector field (P, Q) having a rational first integral of degree 2 which is not polynomial can be written as follows,*

$$\begin{aligned} P &= x(a + bx + dx^2 - ey^2), \\ Q &= y(-a - cy + dx^2 - ey^2), \end{aligned} \tag{1}$$

and its first integral becomes

$$H = \frac{a + bx + cy + dx^2 + ey^2}{xy}. \tag{2}$$

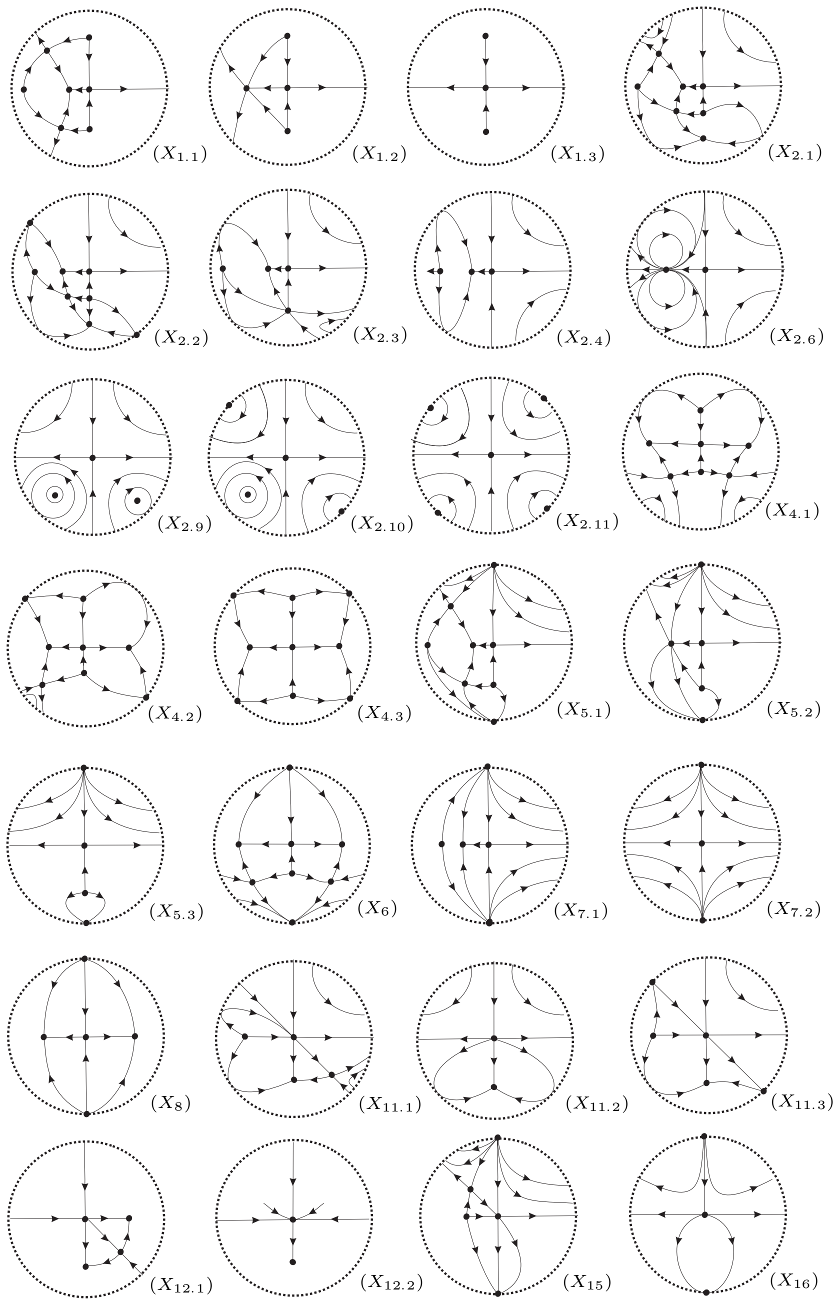

Figure 1. The 28 non-topologically equivalent phase portraits of a planar cubic polynomial vector field of Lotka–Volterra type with a rational first integral of degree 2.

Proof. Assume that $\overline{H} = f/g$ is a rational first integral of degree 2 for a cubic Lotka–Volterra system having no polynomial first integrals. Then f and g are coprime, and the maximum degree of f and g is 2. Since the inverse of a rational first integral is another rational first integral we can assume that g is of degree 2.

Clearly all the orbits of the cubic Lotka–Volterra system having $\overline{H}$ as a first integral are contained in the algebraic curves $f(x, y) - hg(x, y) = 0$ with $h \in \mathbb{R}$ or in $g(x, y) = 0$. Note that this last algebraic curve corresponds to the value infinity of the first integral $\overline{H}$, or to the value zero of the first integral $1/\overline{H}$.

First assume that $g(0, 0) \neq 0$. Then let $h = \overline{H}(0, 0)$, eventually h can be zero. Since $\overline{H}$ is a first integral, the conic $f(x, y) - hg(x, y) = 0$ passing through the origin is formed by solutions of the cubic Lotka–Volterra system. Since this system has the straight lines $x = 0$ and $y = 0$ invariant and both pass through the origin, it follows that the unique conic formed by solutions and containing the origin is $kxy = 0$, where k is a nonzero constant. Therefore $f(x, y) - hg(x, y) = kxy$ for some constant $k \neq 0$. So we can consider the first integral

$$\overline{H} - h = \frac{f(x, y)}{g(x, y)} - h = \frac{f(x, y) - hg(x, y)}{g(x, y)} = k\frac{xy}{g(x, y)}.$$

Since $k/(\overline{H} - h) = g(x, y)/(xy) = (a + bx + cy + dx^2 + ey^2 + mxy)/(xy)$ is also a first integral, we have that the cubic Lotka–Volterra system has the rational function $H = k/(\overline{H} - h) - m$, i.e. the function (2) as a first integral.

Now assume that $g(0, 0) = 0$. Then by the previous arguments we have that $g(x, y) = kxy$ for some nonzero constant k. Since $\overline{H}/k = f(x, y)/(xy) = (a + bx + cy + dx^2 + ey^2 + mxy)/(xy)$ is also a first integral, we have that the cubic Lotka–Volterra system has the rational function $H = \overline{H}/k - m$, i.e. the function (2) as a first integral.

Clearly the Hamiltonian system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial H}{\partial y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial H}{\partial x}$$

has H as a first integral. Then all systems having H as a first integral are of the form

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -F(x, y)\frac{\partial H}{\partial y}, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = F(x, y)\frac{\partial H}{\partial x},$$

where $F(x, y)$ is a function. This last system can be obtained from the previous one doing the rescaling of the time variable $\mathrm{d}t = F(x, y)\,\mathrm{d}s$. Hence the cubic Lotka–Volterra system having the first integral (2) is obtained taking $F(x, y) = x^2y^2$. This completes the proof of the proposition. □

3. Basic results

In this section we introduce the basic definitions, notations and results that we will need for the analysis of the local phase portraits of the finite and infinite singular points of the cubic polynomial vector fields of Lotka–Volterra type.

3.1. Singular points

A point $p \in \mathbb{R}^2$ is said to be a *singular point* of the vector field $X = (P, Q)$ if $P(p) = Q(p) = 0$. We recall first some results which hold when P and Q are analytic functions in a neighbourhood of p. As usual P_x denotes the partial derivative of P with respect to the variable x.

If $\Delta = P_x(p)Q_y(p) - P_y(p)Q_x(p)$ and $T = P_x(p) + Q_y(p)$, then the singular point p is said to be *non-degenerate* if $\Delta \neq 0$. Then p is an isolated singular point. Moreover, p is a *saddle* if $\Delta < 0$, a *node* if $T^2 \geqslant 4\Delta > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), a *focus* if $4\Delta > T^2 > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), and either a *weak focus* or a *centre* if $T = 0 < \Delta$; for more details see [2], p 183.

The singular point p is called *hyperbolic* if the two eigenvalues of the Jacobian matrix $DX(p)$ have a nonzero real part. So, the hyperbolic singular points are the non-degenerate ones except the weak focus and the centres.

A degenerate singular point p (i.e. $\Delta = 0$) with $T \neq 0$ is called *semi-hyperbolic*, and p is isolated in the set of all singular points. Now we summarize the results on semi-hyperbolic singular points that we shall need in this paper, for a proof see theorem 65 of [2].

Proposition 3. *Let* $(0, 0)$ *be an isolated point of the vector field* $(F(x, y), y+G(x, y))$, *where F and G are analytic functions in a neighbourhood of the origin starting at least with quadratic terms in the variables x and y. Let* $y = g(x)$ *be the solution of the equation* $y+G(x, y) = 0$ *in a neighbourhood of* $(0, 0)$. *Assume that the development of the function* $f(x) = F(x, g(x))$ *is of the form* $f(x) = \mu x^m + HOT$ *(higher order terms), where* $m \geqslant 2$ *and* $\mu \neq 0$. *When m is odd, then* $(0, 0)$ *is either an unstable node, or a saddle depending on if* $\mu > 0$, *or* $\mu < 0$, *respectively. In the case of the saddle the stable separatrices are tangent to the x-axis. If m is even, then* $(0, 0)$ *is a saddle node, i.e. the singular point is formed by the union of two hyperbolic sectors with one parabolic sector. The stable separatrix is tangent to the positive (respectively negative) x-axis at* $(0, 0)$ *according to* $\mu < 0$ *(respectively* $\mu > 0$*). The two unstable separatrices are tangent to the y-axis at* $(0, 0)$.

The singular points which are non-degenerate or semi-hyperbolic are called *elementary*.

When $\Delta = T = 0$ but the Jacobian matrix at p is not the zero matrix and p is isolated in the set of all singular points, we say that p is *nilpotent*. Now we summarize the results on nilpotent singular points that we shall need. For a proof see [1], or theorems 66 and 67 and the simplified scheme of section 22.3 of [2].

Proposition 4. *Let* $(0, 0)$ *be an isolated singular point of the vector field* $(y + F(x, y), G(x, y))$, *where F and G are analytic functions in a neighbourhood of the origin starting at least with quadratic terms in the variables x and y. Let* $y = f(x)$ *be the solution of the equation* $y + F(x, y) = 0$ *in a neighbourhood of* $(0, 0)$. *Assume that the development of the function* $G(x, f(x))$ *is of the form* $Kx^\kappa + HOT$ *and* $\Phi(x) \equiv (\partial F/\partial x + \partial G/\partial y)(x, f(x)) = Lx^\lambda + HOT$ *with* $K \neq 0, \kappa \geqslant 2$ *and* $\lambda \geqslant 1$. *Then the following statements hold.*

(1) If κ *is even and*

(1.a) $\kappa > 2\lambda + 1$, *then the origin is a saddle node. Moreover the saddle node has one separatrix tangent to the semi-axis* $x < 0$, *and the other two separatrices tangent to the semi-axis* $x > 0$.

(1.b) $\kappa < 2\lambda + 1$ *or* $\Phi \equiv 0$, *then the origin is a cusp, i.e. a singular point formed by the union of two hyperbolic sectors. Moreover, the cusp has two separatrices tangent to the positive x-axis.*

(2) If κ *is odd and* $K > 0$, *then the origin is a saddle. Moreover, the saddle has two separatrices tangent to the semi-axis* $x < 0$, *and the other two tangent to the semi-axis* $x > 0$.

(3) If κ *is odd,* $K < 0$ *and*

(3.a) λ *even,* $\kappa = 2\lambda + 1$ *and* $L^2 + 4K(\lambda + 1) \geqslant 0$, *or* λ *even and* $\kappa > 2\lambda + 1$, *then the origin is a stable (unstable) node if* $L < 0$ $(L > 0)$, *having all the orbits tangent to the x-axis at* $(0, 0)$.

(3.b) λ *odd,* $\kappa = 2\lambda + 1$ *and* $L^2 + 4K(\lambda + 1) \geqslant 0$, *or* λ *odd and* $\kappa > 2\lambda + 1$ *then the origin is an elliptic saddle, i.e. a singular point formed by the union of one hyperbolic sector and one elliptic sector. Moreover, one separatrix of the elliptic saddle is tangent to the semi-axis* $x < 0$, *and the other to the semi-axis* $x > 0$.

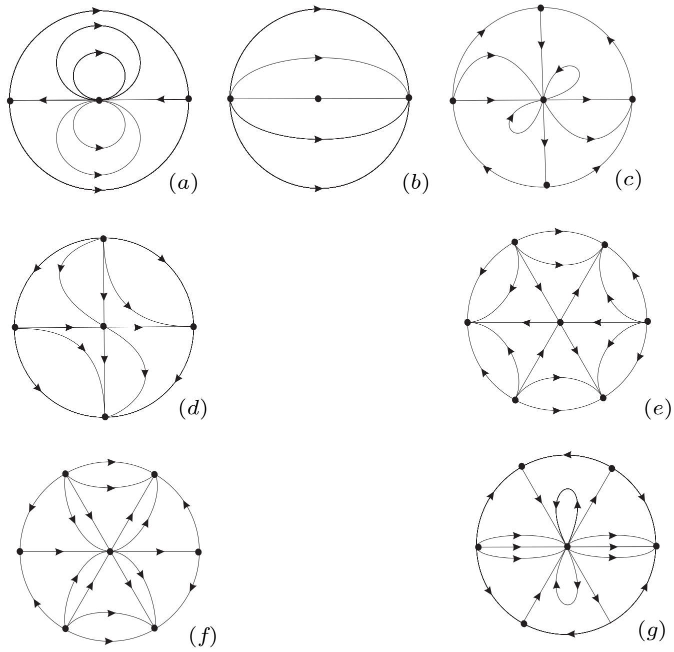

Figure 2. The phase portraits of the homogeneous quadratic vector fields.

(3.c) $\kappa = 2\lambda + 1$ *and* $L^2 + 4K(\lambda + 1) < 0$, *or* $\kappa < 2\lambda + 1$, *then the origin is a focus or a centre, and if* $\Phi(x) \equiv 0$ *then the origin is a centre.*

Finally, if the Jacobian matrix at the singular point p is identically zero, and p is isolated inside the set of all singular points, then we say that p is *linearly zero*. The study of its local phase portrait needs a special treatment (directional blow-ups), see for more details [3]. Note that if in this process, the resulting vector field is a homogeneous quadratic vector field, then the global phase portraits are well known, see figure 2 and for more details [24].

3.2. Poincaré compactification

Let $X \in P_n(\mathbb{R}^2)$ be a planar polynomial vector field of degree n. The *Poincaré compactified vector field* $p(X)$ *corresponding to* X is an analytic vector field induced on $\mathbb{S}^2$ as follows (see, for instance, [14]). Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent plane to $\mathbb{S}^2$ at point y. Identify $\mathbb{R}^2$ with $T_{(0,0,1)}\mathbb{S}^2$. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of X on $\mathbb{S}^2$, one in the northern hemisphere and the other in the southern hemisphere. Denote by X' the vector field $Df \circ X$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. In this paper when we talk about the circle of the infinity of X we simply talk about the infinity.

In order to extend X' to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that X satisfies suitable conditions. In the case that $X \in P_n(\mathbb{R}^2)$, $p(X)$ is the only analytic extension of $y_3^{n-1}X'$ to $\mathbb{S}^2$. On $\mathbb{S}^2\backslash\mathbb{S}^1$ there are two symmetric copies of X, and knowing the behaviour of $p(X)$ around $\mathbb{S}^1$, we know the behaviour of X in a neighbourhood of the infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $p(X)$. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the *Poincaré disc*, and it is denoted by $\mathbb{D}^2$.

In the rest of this paper we say that two polynomial vector fields X and Y on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$.

As $\mathbb{S}^2$ is a differentiable manifold, for computing the expression for $p(X)$, we can consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$; and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively. If we denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so z represents different things according to the local charts under consideration), then some easy computations give for $p(X)$ the following expressions:

$$z_2^n \Delta(z) \left(Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right)\right) \qquad \text{in} \quad U_1, \tag{3}$$

$$\begin{aligned} &z_2^n \Delta(z) \left(P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right)\right) \qquad \text{in} \quad U_2, \\ &\Delta(z)\,(P(z_1, z_2), Q(z_1, z_2)) \qquad \text{in} \quad U_3, \end{aligned} \tag{4}$$

where $\Delta(z) = \left(z_1^2 + z_2^2 + 1\right)^{-\frac{1}{2}(n-1)}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $z_2 = 0$ always denotes the points of $\mathbb{S}^1$. *In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(X)$.* Thus, the expression of $p(X)$ becomes a polynomial vector field in each local chart.

4. Normal forms

Now we shall reduce the number of five parameters of the cubic polynomial vector fields having a rational first integral of degree 2 to at most two parameters.

Proposition 5. *Any cubic polynomial vector field (P, Q) of Lotka–Volterra type having a rational first integral of degree 2 with P and Q coprime can be written as one of the following vector fields:*

$$\begin{aligned}
X_1 &= (x(1 + bx + x^2 + y^2), y(-1 - cy + x^2 + y^2)),\\
X_2 &= (x(1 + bx + x^2 - y^2), y(-1 - cy + x^2 - y^2)),\\
X_3 &= (x(1 + bx - x^2 - y^2), y(-1 - cy - x^2 - y^2)),\\
X_4 &= (x(1 + bx - x^2 + y^2), y(-1 - cy - x^2 + y^2)),\\
X_5 &= (x(1 + bx + x^2), y(-1 - y + x^2)),\\
X_6 &= (x(1 + bx - x^2), y(-1 - y - x^2)),\\
X_7 &= (x(1 + bx + x^2), y(-1 + x^2)),\\
X_8 &= (x(1 + bx - x^2), y(-1 - x^2)),\\
X_9 &= (x(1 + x^2), y(-1 - cy + x^2)),\\
X_{10} &= (x(1 - x^2), y(-1 - cy - x^2)),\\
X_{11} &= (x(bx + x^2 - y^2), y(-y + x^2 - y^2)),\\
X_{12} &= (x(bx - x^2 - y^2), y(-y - x^2 - y^2)),\\
X_{13} &= (x(x + x^2 - y^2), y(x^2 - y^2)),\\
X_{14} &= (x(x - x^2 - y^2), y(-x^2 - y^2)),
\end{aligned}$$

$$X_{15} = (x(x + x^2), y(-y + x^2)),$$
$$X_{16} = (x^3, y(-y + x^2)),$$

with $b \geqslant 0$ *and* $c \geqslant 0$.

Proof. Let $\bar{x} = \alpha x$, $\bar{y} = \beta y$, $\bar{t} = \gamma t$. Then the transformed vector field $(\bar{P}, \bar{Q})$, writes in the transformed variables as $(\bar{x}(a/\gamma + b\bar{x}/(\alpha\gamma) + d\bar{x}^2/(\alpha^2\gamma) - e\bar{y}^2/(\beta^2\gamma)), (\bar{y}(-a/\gamma - c\bar{y}/(\beta\gamma) + d\bar{x}^2/(\alpha^2\gamma) - e\bar{y}^2/(\beta^2\gamma))$.

Case 1: $ade \neq 0$. Then we can choose α, β and γ conveniently for obtaining X_k with $k = 1, 2, 3, 4$.

Case 2: $e = 0$. Consequently $d \neq 0$, otherwise the vector field should not be cubic.

Subcase 2.1: $ac \neq 0$. Choosing α, β and γ conveniently we get the vector fields X_k with $k = 5, 6$.

Subcase 2.2: $c = 0$ and $ab \neq 0$. We obtain the vector fields X_k with $k = 7, 8$.

Subcase 2.3: $b = 0$ and $ac \neq 0$. We obtain the vector fields X_k with $k = 9, 10$.

Subcase 2.4: $a = 0$ and $c \neq 0$. We get the vector fields X_k with $k = 15, 16$.

Subcase 2.5: $a = c = 0$. In this subcase the polynomials P and Q are not coprime.

Case 3: $e \neq 0$ and $a = 0$.

Subcase 3.1: $d \neq 0$. Again choosing α, β and γ conveniently we get the vector fields X_k with $k = 11, 12, 13, 14$.

Subcase 3.2: $d = 0$. In this subcase we again obtain the vector fields X_k with $k = 15, 16$ doing the change of variables $(x, y, t) \mapsto (y, x, -t)$.

Case 4: $ea \neq 0$ and $d = 0$. In this subcase we again obtain the vector fields X_k with $k = 5, 6$ doing the change of variables $(x, y, t) \mapsto (y, x, -t)$.

Finally we note that if we do the change of variables $(x, y) \to (-x, y)$, we get another vector field of the form (1) with $-b$ instead of b. So, we can assume that $b \geqslant 0$ and, similarly, we also can assume that $c \geqslant 0$. This completes the proof of the proposition. □

Note that the families from X_1 to X_4 depend on two parameters b and c, the families from X_5 to X_{12} depend only on one parameter b, and the families from X_{13} to X_{16} are reduced to a unique vector field.

5. The infinite singular points

In this section we use the definitions and relations introduced in subsection 3.2 dedicated to the Poincaré compactification. We first study the chart U_1. As we see in the next result the infinity in the Poincaré compactification is filled of singular points.

Proposition 6. *The vector fields 12 have the infinite filled with singular points.*

Proof. The vector field in the local chart U_1 is

$$\left(-z_1 z_2(b + cz_1 + 2az_2), -z_2\left(d + bz_2 - ez_1^2 + az_2^2\right)\right). \tag{5}$$

Therefore the infinity, $z_2 = 0$ is a line of singular points. □

Even that all the infinity is full of singular points, there are special ones which have interest in building our phase portraits. As a matter of fact, we note that the vector field (5), after rescaling the time by z_2, writes $\big(-z_1(b+cz_1+2az_2), -\big(d+bz_2-ez_1^2+az_2^2\big)\big)$. Now we study the infinite singular points of this new vector field. We do not consider the case $d=0$, because it does not occur in our normal forms. If $bcd \neq 0$ and $c^2d-b^2e=0$, then there is the additional singular point $(-b/c, 0)$ in U_1 whose eigenvalues are $\pm(b/c)\sqrt{c^2-4ae}$. If $d \neq 0$ and $b=c=0$ then there are two singular points at infinity in U_1, namely $(\pm\sqrt{d/e}, 0)$ if $de>0$.

In the chart U_2 and after rescaling the time by z_2, the vector field writes $\big(z_1(c+bz_1+2az_2), \big(e+cz_2-dz_1^2+az_2^2\big)\big)$. If $e=0$ then the origin of the chart U_2 is a singular point with both eigenvalues equal to c. Consequently, if $e=0$ and $c>0$, then the origin of U_2 is an unstable node; and if $e=c=0$ then the origin of U_2 is linearly zero, and the system becomes the homogeneous quadratic vector field $\big(z_1(bz_1+2az_2), -dz_1^2+az_2^2\big)$.

Proposition 7. *The following statements hold.*

(a) $X_1, X_3, X_{12}, X_{13}, X_{14}$ *have no infinite singular points.*
(b) X_2 *has only the infinite singular point* $(-1,0)$ *in* U_1 *if* $b=c\neq 0$*. This singular point is a saddle if* $b>2$*, and a centre if* $0<b<2$*. If* $b=0$ *then there are two singular points* $(\pm 1, 0)$ *in* U_1*, which are centres. If* $b=2$ *then the components of* X_2 *are not coprime.*
(c) X_4 *has only the infinite singular point* $(-1,0)$ *in* U_1 *if* $b=c\neq 0$*, which is a saddle. If* $b=c=0$*, there are two saddles in* U_1*.*
(d) X_k *with* $k\in\{5,6\}$ *has only the origin of* U_2 *as an infinite singular point, which is an unstable node.*
(e) X_7 *has only the origin of* U_2 *as infinite singular point, which is linearly zero. If* $b>2$*, then its local phase portrait is the origin of figure* 2(*f*))*. If* $b<2$*, its local phase portrait is the origin of figure* 2(*a*))*. If* $b=2$ *the components of* X_7 *are not coprime.*
(f) X_8 *has only the origin of* U_2 *as an infinite singular point, which is linearly zero and its local phase portrait is the origin of figure* 2(*f*))*.*
(g) X_9 *has only the origin of* U_2 *as an infinite singular point, which is a hyperbolic unstable node except if* $c=0$ *where it is linearly zero. Its local phase portrait is the origin of figure* 2(*a*))*.*
(h) X_{10} *has only the origin of* U_2 *as an infinite singular point, which is a hyperbolic unstable node except if* $c=0$ *where it is linearly zero and its local phase portrait is the origin of figure* 2(*f*))*.*
(i) X_{11} *has only the infinite singular point* $(-1,0)$ *in* U_1 *if* $b=1$*, which is a saddle.*
(j) X_{15} *and* X_{16} *have only the origin of* U_2 *as an infinite singular point, which is an unstable node.*

Proof. The proof of statements (a)–(d) and (i) and (j) except the centre for statement (b), which is studied here, follow easily from the previous information on the singular points at infinity in the local charts U_1 and U_2.

In order to show that $(-1,0)$ of the chart U_1 of X_2 is a centre when $b=c$, we must transform the vector field into the canonical form for a vector field to be candidate to have a centre. We follow the next steps.

The rescaled vector field of the chart U_1 of X_2 is $\big(-z_1(b+bz_1+2z_2), -\big(1+bz_2-z_1^2+z_2^2\big)\big)$. Translating the singular point to the origin, the vector field writes $\big(bz_1+2z_2-2z_1z_2-bz_1^2, -2z_1-bz_2+z_1^2-z_2^2\big)$, where we have done the same name to the new variables z_1 and z_2. At $z_1=z_2=0$ the Jacobian matrix is

$$M=\begin{pmatrix} b & 2 \\ -2 & -b \end{pmatrix},$$

with eigenvalues $\pm\sqrt{b^2-4}$. So, the Jordan matrix is

$$J = \begin{pmatrix} 0 & -\sqrt{4-b^2} \\ \sqrt{4-b^2} & 0 \end{pmatrix}$$

and we can find easily from $M = BJB^{-1}$ a transformation matrix B

$$B = \begin{pmatrix} -\frac{b}{2} & -\frac{\sqrt{4-b^2}}{2} \\ 1 & 0 \end{pmatrix}$$

such that it will allow us to build the canonical vector field through the change of variables

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = B^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The canonical vector field writes $\left(-Z_2 + RZ_1^2 - bZ_1Z_2/2 + S, Z_1 + bZ_1^2/4 - TZ_1Z_2 - bZ_2^2/4\right)$, where $R = -\sqrt{4-b^2}/4$, $S = \sqrt{4-b^2}/4$ and $T = \sqrt{4-b^2}/2$. Consequently as $R + S = 0$, using [4] we can say that the singular point $(-1, 0)$ of the chart U_1 is a centre when $b = c$.

The vector field X_7 in the chart U_2 is $\left(z_1(bz_1 + 2z_2), -z_1^2 + z_2^2\right)$, so it is quadratic and homogeneous. As a consequence its local phase portrait is the origin of one of the phase portraits of figure 2. Now, in order to find the corresponding one, we must study the infinity. When $b > 2$ there are two real singular points in the chart U_1 namely $(-b + \sqrt{b^2-4})/2, 0)$ and $(-b - \sqrt{b^2-4})/2, 0)$, having both eigenvalues equal to $\sqrt{b^2-4}$ and $-\sqrt{b^2-4}$ respectively. Moreover, the origin of the chart U_2 is a saddle. This proves that the local phase portrait of X_7 at the origin of U_2 is the one of figure 2(f) if $b > 2$, and of figure 2(a) if $b < 2$.

The expression of X_8 in the chart U_2 is $\left(z_1(bz_1 + 2z_2), z_1^2 + z_2^2\right)$, which is quadratic and homogeneous. As before its local phase portrait is the origin of one of the phase portraits of figure 2 and in order to find the corresponding one, we must study the infinity. Then in the chart U_1 we obtain two singular points namely $(-b + \sqrt{b^2+4})/2, 0)$ and $(-b - \sqrt{b^2+4})/2, 0)$, having both eigenvalues equal to $\sqrt{b^2+4}$ and $-\sqrt{b^2+4}$ respectively. Moreover, the origin of the chart U_2 is a saddle. This shows that the local phase portrait of X_8 at the origin of U_2 is the one of figure 2(f).

The vector field X_9 in the chart U_2 is $\left(z_1(2z_2 + c), cz_2 - z_1^2 + z_2^2\right)$. The origin has eigenvalues (c, c) and consequently when $c = 0$ is linearly zero. But now the vector field is quadratic and homogeneous and as before we can use the results given in figure 2. In order to find the corresponding phase portrait, we must study the infinity. The are no real singular points in the chart U_1. The origin of the chart U_2 is a saddle. This proves that if $c = 0$, the local phase portrait of X_9 at the origin of U_2 is the one of figure 2(a).

The expression of X_{10} in the chart U_2 is $\left(z_1(2z_2 + c), cz_2 - z_1^2 + z_2^2\right)$. The origin has eigenvalues (c, c) and consequently when $c = 0$ is linearly zero. But now the vector field is quadratic and homogeneous and as before we can use the results given in figure 2. In order to find the corresponding phase portrait, we must study the infinity. Then in the chart U_1 we obtain two singular points namely $(\pm 1, 0)$ which are stable and unstable nodes respectively. Moreover, the origin of the chart U_2 is a saddle. This proves that if $c = 0$, the local phase portrait of X_{10} at the origin of U_2 is the one of figure 2(f). □

6. The finite singular points

The vector field X_1. For this vector field we define $p = b^2 - 4$ and $q = c^2 + 4$. Now using the results of subsection 3.1, we study the local phase portraits of its singular points.

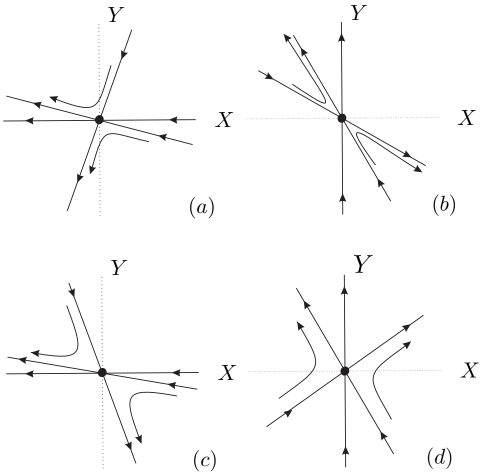

Figure 3. The local phase portrait of the vector fields (6), (7), (9) for $c > 2$ and (11) at the origin.

(1.1) If $b > 2$ and $c \geqslant 0$, then X_1 has the following seven finite singular points: $(0, 0)$ is a hyperbolic saddle; $(0, (c \pm \sqrt{q})/2)$ are hyperbolic unstable nodes; $((-b + \sqrt{p})/2, 0)$ is a hyperbolic stable node, and $((-b - \sqrt{p})/2, 0)$ is a hyperbolic unstable node and $(-2\sqrt{q}/(b\sqrt{q} \pm c\sqrt{p}), (cp \mp b\sqrt{pq})/2(b^2 + c^2))$ are hyperbolic saddles.

(1.2) If $b = 2$ and $c \geqslant 0$, then X_1 has the following four finite singular points: $(0, 0)$ is a hyperbolic saddle; $(0, (c \pm \sqrt{q})/2)$ are hyperbolic unstable nodes and $(-1, 0)$ is linearly zero. To study this singular point, we first translate it to the origin; i.e. we consider the change of variables $X = x + 1$, $Y = y$, and the vector field X_1 becomes

$$X_1 = (-X^2 - Y^2 + XY^2 + X^3, -2XY - cY^2 + X^2Y + Y^3). \tag{6}$$

Now we do the blow-up $(X, Y) \to (X, v)$, where $Y = vX$, and the vector field (6) becomes

$$X_1 = (-X^2 - X^2v^2 + X^3 + X^3v^2, -Xv - cXv^2 + Xv^3).$$

Rescaling the independent variable by X we get the vector field

$$X_1 = (-X - Xv^2 + X^2 + X^2v^2, -v - cv^2 + v^3).$$

On the straight line $X = 0$ there are three singular points, namely $(0, 0)$, $(0, (c \pm \sqrt{(q)}/2))$. The first is a hyperbolic stable node and the other two are saddles. Undoing the blow-up, the origin of the vector field (6) has the local phase portrait of figure 3(a).

(1.3) If $0 \leqslant b < 2$ and $c \geqslant 0$, then X_1 has the following three finite singular points: $(0, 0)$ is a hyperbolic saddle and $(0, (c \pm \sqrt{q})/2)$ are hyperbolic unstable nodes.

Now we know the local phase portrait at the finite and infinite singular points of X_1. Using the rational first integral we determine the global phase portrait in the Poncaré disc, see the next result.

Proposition 8. *The phase portrait of X_1 is topologically equivalent to*

(X1.1) Figure 1$(X_{1.1})$ *if* $b > 2$ *and* $c \geqslant 0$*;*
(X1.2) Figure 1$(X_{1.2})$ *if* $b = 2$ *and* $c \geqslant 0$*;*
(X1.3) Figure 1$(X_{1.3})$ *if* $0 \leqslant b < 2$ *and* $c \geqslant 0$.

The vector field X_2. For the vector field X_2 we define $p = b^2 - 4$ and $q = c^2 - 4$. Following the preceding analysis, we obtain for the singular points of the vector field X_2.

(2.1) If $b > 2, c > 2$ and $b \neq c$ then X_2 has the following seven finite singular points: $(0, 0)$ is a hyperbolic saddle; $(0, (-c + \sqrt{q})/2, 0)$ is a hyperbolic unstable node, and $(0, (-c - \sqrt{q})/2, 0)$ is a hyperbolic unstable node; $((-b + \sqrt{p})/2, 0)$ is a hyperbolic stable node, and $((-b - \sqrt{p})/2, 0)$ is a hyperbolic unstable node and $(-2q/(bq \pm c\sqrt{pq}),$ $(-cp \pm b\sqrt{pq})/2(b^2 - c^2))$ are hyperbolic saddles.

(2.2) If $b = c > 2$, then X_2 has the following six finite singular points: $(0, 0)$ is a hyperbolic saddle; $(0, (-b + \sqrt{p})/2)$ is a hyperbolic unstable node, and $(0, (-b - \sqrt{p})/2)$ is a hyperbolic stable node; $((-b + \sqrt{p})/2, 0)$ is a hyperbolic stable node, and $((-b - \sqrt{p})/2, 0)$ is a hyperbolic unstable node and $-(b^{-1}, b^{-1})$ is a hyperbolic saddle.

(2.3) If $b > 2$ and $c = 2$, then X_2 has the following four finite singular points: $(0, 0)$ is a hyperbolic saddle; $((-b + \sqrt{p})/2, 0)$ is a hyperbolic stable node, and $((-b - \sqrt{p})/2, 0)$ is a hyperbolic unstable node; $(0, -1)$ is linearly zero. To study this singular point, we first translate it to the origin; i.e. we consider the change of variables $X = x + 1$, $Y = y$, and the vector field X_2 becomes

$$X_2 \quad = \quad (bX^2 + 2XY - XY^2 + X^3, -X^2 + Y^2 + X^2Y - Y^3), \tag{7}$$

Now we do the blow-up $(X, Y) \to (X, v)$, where $Y = vX$, and the vector field (7) becomes

$$X_2 = (bX^2 + 2X^2v + X^3 - X^3v^2, -X - bXv - Xv^2).$$

Rescaling the independent variable by X we get the vector field

$$X_2 = (bX + 2Xv + X^2 - X^2v^2, -1 - bv - v^2). \tag{8}$$

On the straight line $X = 0$ there are three singular points, namely $(0, 0)$, $(0, (b \pm \sqrt(p)/2))$, The first is a hyperbolic stable node and the other two are saddles. Undoing the blow-up, the origin of the vector field (7) has the local phase portrait of figure 3(*b*).

(2.4) If $b > 2$ and $0 \leqslant c < 2$, then X_2 has the following three finite singular points: $(0, 0)$ is a hyperbolic saddle; $((-b + \sqrt{p})/2, 0)$ is a hyperbolic stable node; and $((-b - \sqrt{p})/2, 0)$ is a hyperbolic unstable node.

(2.5) If $b = 2$ and $c > 2$, then X_2 has the following four finite singular points: $(0, 0)$ is a hyperbolic saddle; $(0, (-c + \sqrt{q})/2)$ is a hyperbolic unstable node, and $(0, (-c - \sqrt{q})/2)$ is a hyperbolic stable node and $(-1, 0)$ is linearly zero. To study this singular point, we first translate it to the origin; i.e. we consider the change of variables $X = x + 1$, $Y = y$, and the vector field X_2 becomes

$$X_2 = (-X^2 + Y^2 - XY^2 + X^3, -2XY - cY^2 + X^2Y - Y^3). \tag{9}$$

Now doing the blow-up $(X, Y) \to (X, v)$, where $Y = vX$, the vector field (9) transforms into

$$X_2 = (-X^2 + X^2v^2 + X^3 - X^3v^2, -Xv - cXv^2 - Xv^3).$$

Rescaling the independent variable by X we get the vector field

$$X_2 = (-X + Xv^2 + X^2 - X^2v^2, -v - cv^2 - v^3). \tag{10}$$

On the straight line $X = 0$ there are three singular points, namely $(0, 0)$, $(0, -c \pm \sqrt{q}/2)$. The first is a stable node and the other two are saddles. Undoing the blow-up, the origin of the vector field (9) has the local phase portrait of figure 3(*c*).

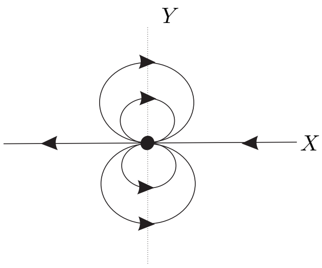

Figure 4. The local phase portrait of the vector field (9) for $0 \leqslant c < 2$ at the origin.

(2.6) If $b = 2$ and $0 \leqslant c < 2$, then X_2 has the following two finite singular points: $(0, 0)$ is a hyperbolic saddle and $(0, -1)$ is linearly zero. To study this singular point, we follow the steps of case 2.5. It follows that now the vector field (10) has only the origin as a real singular point. Undoing the blow-up, we obtain that the origin of the vector field (9) has the local phase portrait of figure 4.

(2.7) If $0 \leqslant b < 2$ and $c > 2$, then X_2 has the following three singular points: $(0, 0)$ is a hyperbolic saddle; $(0, (-c + \sqrt{q})/2)$ is a hyperbolic unstable node; and $(0, (-c - \sqrt{q})/2)$ is a hyperbolic stable node.

(2.8) If $0 \leqslant b < 2, c = 2$, then X_2 has the following two finite singular points: $(0, 0)$ is a hyperbolic saddle and $(0, -1)$ is linearly zero. To study this singular point, we follow the steps of case 2.3. It follows that now the vector field (8) has only the origin as a real singular point. Undoing the blow-up, we obtain that the origin of the vector field (7) has the local phase portrait of figure 4.

(2.9) If $0 \leqslant b < 2, 0 \leqslant c < 2$ and $b \neq c$ then X_2 has the following three finite singular points: $(0, 0)$ is a hyperbolic saddle and $(-2q/(bq \pm c\sqrt{pq}), (-cp \pm b\sqrt{pq})/2(b^2 - c^2))$ are centres.

(2.10) If $0 < b < 2, 0 < c < 2$ and $b = c$, then the origin is the unique finite singular point, which is a centre.

(2.11) If $b = c = 0$, then the origin is the unique finite singular point, which is a centre.

Note that if $b = c = 2$ then X_2 is not coprime.

Summarizing we have

Proposition 9. *The phase portrait of X_2 is topologically equivalent to*

(X2.1) Figure 1$(X_{2.1})$ if $b > 2, c > 2$ and $b \neq c$;
(X2.2) Figure 1$(X_{2.2})$ if $b = c > 2$;
(X2.3) Figure 1$(X_{2.3})$ if $b > 2$ and $c = 2$;
(X2.4) Figure 1$(X_{2.4})$ if $b > 2$ and $0 \leqslant c < 2$;
(X2.5) Figure 1$(X_{2.3})$ if $b = 2$ and $c > 2$;
(X2.6) Figure 1$(X_{2.6})$ if $b = 2$ and $0 \leqslant c < 2$;
(X2.7) Figure 1$(X_{2.4})$ if $0 \leqslant b < 2$ and $c > 2$;
(X2.8) Figure 1$(X_{2.6})$ if $0 \leqslant b < 2$ and $c = 2$;
(X2.9) Figure 1$(X_{2.9})$ if $0 \leqslant b < 2, 0 \leqslant c < 2$ and $b \neq c$.
(X2.10) Figure 1$(X_{2.10})$ if $0 < b = c < 2$.
(X2.11) Figure 1$(X_{2.11})$ if $b = c = 0$.

The vector field X_3. For the vector field X_3 we define $p = b^2 + 4$ and $q = c^2 - 4$. Following the preceding analysis, we obtain for the singular points of the vector field X_3.

(3.1) If $b \geqslant 0$ and $c > 2$, then X_3 has the following seven finite singular points: $(0, 0)$ is a hyperbolic saddle; $(0, (-c + \sqrt{q})/2)$ is a hyperbolic unstable node, and $(0, (-c - \sqrt{q})/2)$ is a hyperbolic stable node; $((b \pm \sqrt{p})/2, 0)$ are hyperbolic stable nodes and $(-2\sqrt{q}/(b\sqrt{q} \pm c\sqrt{p}), (cp \mp b\sqrt{pq})/2(b^2 - c^2))$ are hyperbolic saddles.

(3.2) If $b \geqslant 0$ and $c = 2$, then X_3 has the following four finite singular points: $(0, 0)$ is a hyperbolic saddle; $((b \pm \sqrt{p})/2, 0)$ are hyperbolic stable nodes; $(0, -1)$ is linearly zero. To study this singular point, we first translate it to the origin; i.e. we consider the change of variables $X = x$, $Y = y + 1$, and the vector field X_3 becomes

$$X_3 = (2XY + bX^2 - XY^2 - X^3, X^2 + Y^2 - X^2Y - Y^3). \tag{11}$$

Now we do the blow-up $(X, Y) \to (X, v)$, where $Y = vX$, and the vector field (11) becomes

$$X_3 = (bX^2 - X^3 + 2X^2v - X^3v^2, X - bXv - Xv^2).$$

Rescaling the independent variable by X we get the vector field

$$X_3 = (bX + 2Xv - X^2 - X^2v^2, 1 - bv - v^2).$$

On the straight line $X = 0$ there are two singular points, namely $(0, (b \pm \sqrt{(p)}/2))$, which are saddles. Undoing the blow-up, the origin of the vector field (11) has the local phase portrait of figure 3(*d*).

(3.3) If $b \geqslant 0$ and $0 \leqslant c < 2$, then X_3 has the following three finite singular points: $(0, 0)$ is a hyperbolic saddle and $((b \pm \sqrt{p})/2, 0)$ are hyperbolic stable nodes.

Summarizing we have

Proposition 10. *The phase portrait of* X_3 *is topologically equivalent to*

(X3.1) Figure 1$(X_{1.1})$ *if* $b \geqslant 0, c \geqslant 0$ *and* $b \neq c$*;*
(X3.2) Figure 1$(X_{1.2})$ *if* $b \geqslant 0$ *and* $c = 2$*;*
(X3.3) Figure 1$(X_{1.3})$ *if* $b \geqslant 0$ *and* $0 \leqslant c < 2$.

The vector field X_4. For the vector field X_4 we define $p = b^2 + 4$ and $q = c^2 + 4$. Following the preceding analysis, we obtain for the singular points of the vector field X_4.

(4.1) If $b \geqslant 0, c \geqslant 0$ and $b \neq c$ then X_4 has the following seven finite singular points: $(0, 0)$ is a hyperbolic saddle; $(0, (c \pm \sqrt{q})/2)$ are hyperbolic unstable nodes; $((b \pm \sqrt{p})/2, 0)$ are hyperbolic stable nodes and $(-2\sqrt{q}/(b\sqrt{q} \pm c\sqrt{p}), (cp \mp b\sqrt{pq})/2(b^2 - c^2))$ are hyperbolic saddles.

(4.2) If $b = c \neq 0$, then X_4 has six finite singular points, namely $(0, 0)$ is a hyperbolic saddle; $(0, (c \pm \sqrt{q})/2)$ are hyperbolic unstable nodes; $((b \pm \sqrt{p})/2, 0)$ are hyperbolic stable nodes and $-(b^{-1}, b^{-1})$ is a hyperbolic saddle.

(4.3) If $b = c = 0$, then X_4 has five finite singular points, namely $(0, 0)$ is a hyperbolic saddle; $(0, \pm 1)$ are hyperbolic unstable nodes and $(\pm 1, 0)$ are hyperbolic stable nodes.

Summarizing we have

Proposition 11. *The phase portrait of* X_4 *is topologically equivalent to*

(X4.1) Figure 1$(X_{4.1})$ *if* $b \geqslant 0$ *and* $c \geqslant 0$*;*
(X4.2) Figure 1$(X_{4.2})$ *if* $b = c \neq 0$*;*
(X4.3) Figure 1$(X_{4.3})$ *if* $b = c = 0$.

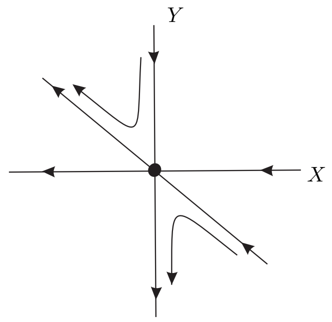

Figure 5. The local phase portrait of the vector field (12) at the origin.

The vector field X_5. For the vector field X_5 we define $p_- = b^2 - 4$. Following the preceding analysis, we obtain for the singular points of the vector field X_5.

(5.1) If $b > 2$, then X_5 has six finite singular points, namely (0, 0) is a hyperbolic saddle; (0, −1) is a hyperbolic unstable node; $\left(-\frac{1}{2}(b - \sqrt{p_-}), 0\right)$ is a stable node and $\left(-\frac{1}{2}(b + \sqrt{p_-}), 0\right)$ is an unstable node and $\left(-\frac{1}{2}(b \pm \sqrt{p_-}), \frac{1}{2}(p_- \pm b\sqrt{p_-})\right)$ are hyperbolic saddles.

(5.2) If $b = 2$, then X_5 has three finite singular points, namely (0, 0) is a hyperbolic saddle; (0, −1) is a hyperbolic unstable node; (−1, 0) is linearly zero. To study this singular point, we first translate it to the origin; i.e. we consider the change of variables $X = x + 1$, $Y = y$, and the vector field X_5 becomes

$$X_5 = (-X^2 + X^3, -2XY - Y^2 + X^2Y). \tag{12}$$

Now we do the blow-up $(X, Y) \to (X, v)$, where $Y = vX$, and the vector field (12) becomes

$$X_5 = (-X^2 + X^3, -Xv - X^2v - Xv^2 + X^2v^2).$$

Rescaling the independent variable by X we get the vector field

$$X_5 = (-X + X^2, -v - Xv - v^2 + Xv^2).$$

On the straight line $X = 0$ there are three singular points, namely (0, 0), (0, −1). The first is a hyperbolic stable node and the other two are saddles. Undoing the blow-up, the origin of the vector field (12) has the local phase portrait of figure 5.

(5.3) If $0 \leqslant b < 2$, then X_5 has two finite singular points, namely: (0, 0) is a hyperbolic saddle and (0, −1) is a hyperbolic unstable node.

Summarizing we have

Proposition 12. *The phase portrait of* X_5 *is topologically equivalent to*

(X5.1) Figure 1($X_{5.1}$) *if* $b > 2$*;*
(X5.2) Figure 1($X_{5.2}$) *if* $b = 2$*;*
(X5.3) Figure 1($X_{5.3}$) *if* $0 \leqslant b < 2$*.*

The vector field X_6. For the vector field X_6 we define $p_+ = b^2 + 4$. Following the preceding analysis, we obtain that X_6 has six finite singular points, namely: (0, 0) is a hyperbolic saddle; (0, −1) is a hyperbolic unstable node; $\left(\frac{1}{2}(b \pm \sqrt{p_+}), 0\right)$ are stable nodes and $\left(\frac{1}{2}(b \pm \sqrt{p_+}), \frac{1}{2}(-p_+ \mp b\sqrt{p_+})\right)$ are hyperbolic saddles.

Summarizing we have

Proposition 13. *The phase portrait of X_6 is topologically equivalent to*

(X6) Figure1(X_6).

The vector field X_7. For the vector field X_7, following the preceding analysis, we obtain for the finite singular points:

(7.1) If $b > 2$, then X_7 has three singular points: $(0, 0)$ is a hyperbolic saddle; $\left(-\frac{1}{2}(b + \sqrt{p_-}), 0\right)$ is an unstable node and $\left(-\frac{1}{2}(b - \sqrt{p_-}), 0\right)$ is a stable node.

(7.2) If $0 \leqslant b < 2$, then X_7 has only the origin as a finite singular point, which is a hyperbolic saddle.

Note that $b = 2$ is excluded as X_7 is not coprime.

Summarizing we have

Proposition 14. *The phase portrait of X_7 is topologically equivalent to*

(X7.1) Figure 1$(X_{7.1})$ *if* $b > 2$;
(X7.2) Figure 1$(X_{7.2})$ *if* $0 \leqslant b < 2$.

The vector field X_8. X_8 has three finite singular points, namely $(0, 0)$ is a hyperbolic saddle and $\left(\frac{1}{2}(b \pm \sqrt{p_+}), 0\right)$ are stable nodes.

Summarizing we have

Proposition 15. *The phase portrait of X_8 is topologically equivalent to*

(X8) Figure 1(X_8).

The vector field X_9. Fist we note that for $c = 1$ and $b = 0$, the vector fields X_9 and X_5 coincide. In fact for the vector field X_9, following the preceding analysis, we obtain for the finite singular points:

(9.1) If $c \neq 0$, X_9 has two finite singular points: $(0, 0)$ is a hyperbolic saddle and $(0, -1/c))$ is an unstable node.

From the preceding it is clear that the global phase portraits of $X_{9.1}$ and $X_{5.3}$ are the same.

(9.2) If $c = 0$, then X_9 coincides with $X_{7.2}$ when $b = 0$.

Summarizing we have

Proposition 16. *The phase portrait of $X_{9.1}$ is topologically equivalent to*

(X9.1) Figure 1$(X_{5.3})$ *if* $c \neq 0$;
(X9.2) Figure 1$(X_{7.2})$ *if* $c = 0$.

The vector field X_{10}. Fist we note that for $c = 1$ and $b = 0$, the vector fields X_{10} and X_6 coincide. In fact for the vector field X_{10}, following the preceding analysis, we obtain for the singular points:

(10.1) If $c \neq 0$, then X_{10} has six finite singular points, namely $(0, 0)$ is a hyperbolic saddle; $(0, -1/c)$ is an unstable node; $(\pm 1, 0)$ are stable nodes and $(\pm 1, -2/c)$ are hyperbolic saddles.

(10.2) If $c = 0$, then X_{10} coincides with X_8 when $b = 0$.

Summarizing we have

Proposition 17. *The phase portrait of X_{10} is topologically equivalent to*

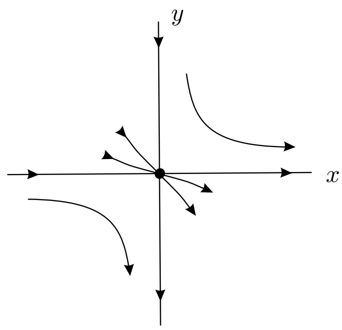

Figure 6. The local phase portrait of the vector fields $X_{11.1}$, $X_{11.3}$ and X_{15} at the origin.

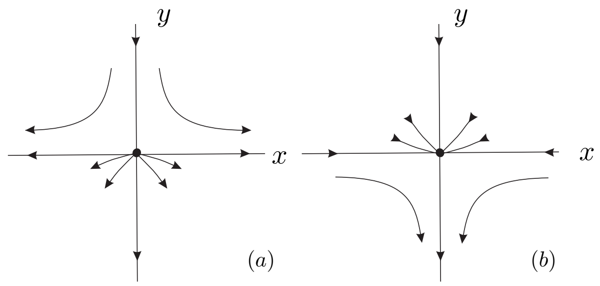

Figure 7. The local phase portrait of the vector fields $X_{11.2}$, $X_{12.2}$ and X_{16} at the origin.

(X10.1) Figure 1(X_6) *if* $c \neq 0$*;*
(X10.2) Figure 1(X_8) *if* $c = 0$.

The vector field X_{11}. Following the preceding analysis, we obtain for the singular points of the vector field X_{11}.

(11.1) If $0 < b < 1$, or $b > 1$, then X_{11} has four singular points, namely $(-b, 0)$ is a unstable node; $(0, -1)$ is a stable node; $(b/(b^2 - 1), -b^2/(b^2 - 1))$ is a saddle and $(0, 0)$ is linearly zero. To study this singular point we do the blow-up $(x, y) \to (x, v)$, where $y = vx$, and the vector field X_{11} becomes $X_{11} = (bx^2 + x^3 - x^3v^2, -bxv - xv^2)$. Rescaling the independent variable by x we get the vector field $X_{11} = (bx + x^2 - x^2v^2, -bv + v^2)$. If $b \neq 0$, on the straight line $x = 0$ there are two singular points, namely $(0, 0)$ which is a saddle and $(0, b)$ which is an unstable node. Undoing the blow-up, the origin of the vector field X_{11} has the local phase portrait of figure 6.

(11.2) If $b = 0$, then X_{11} has two singular points, namely $(0, 1)$ is a stable node and $(0, 0)$ is linearly zero. To study this singular point we follow the same development as for (11.1), but now only the origin of the straight line $x = 0$ is a singular point and the vector field X_{11} has the local phase portrait of figure 7(a).

(11.3) If $b = 1$, then X_{11} has three singular points, namely $(-b, 0)$ is an unstable node; $(0, 1)$ is a stable node; $(0, 0)$ has the same behaviour as the origin of (11.1).

Summarizing we have

Proposition 18. *The phase portrait of X_{11} is topologically equivalent to*

(X11.1) Figure 1$(X_{11.1})$ *if* $0 < b < 1$ *or* $b > 1$*;*
(X11.2) Figure 1$(X_{11.2})$ *if* $b = 0$*;*
(X11.3) Figure 1$(X_{11.3})$ *if* $b = 1$.

The vector field X_{12}. Following the preceding analysis, we obtain for the singular points of the vector field X_{12}.

(12.1) If $b \neq 0$, then X_{12} has four singular points: $(b, 0)$ is a stable node; $(0, -1)$ is a stable node; $b/(b^2+1), -b^2/(b^2+1))$ is a saddle; $(0, 0)$ is linearly zero. To study this singular point we do the blow-up $(x, y) \to (x, v)$, where $y = vx$, and the vector field X_{12} becomes

$$X_{12} = (bx^2 - x^3 - x^3v^2, -bxv + xv^2).$$

Rescaling the independent variable by x we get the vector field

$$X_{12} = (bx - x^2 - x^2v^2, -bv + v^2).$$

On the straight line $x = 0$ there are two singular points, namely $(0, 0)$ which is a saddle and $(0, b)$ which is an unstable node. Undoing the blow-up, the origin of the vector field X_{12} has the local phase portrait of figure 6.

(12.2) If $b = 0$, then X_{12} has two singular points: $(0, -1)$ is a saddle and $(0, 0)$ is linearly zero. To study this singular point we do the blow-up $(x, y) \to (x, v)$, where $y = xv$, and the vector field X_{12} becomes $X_{12} = (-x^3 - x^3v^2, -xv^2)$. Rescaling the independent variable by x we get the vector field $X_{12} = (-x^2 - x^2v^2, -v^2)$. On the straight line $x = 0$ the origin is the unique singular point, which is linearly zero. To study this singular point we do the second blow-up $(x, v) \to (x, w)$, where $v = xw$, and after rescaling the independent variable by x, the vector field X_{12} becomes $X_{12} = (-x - x^3w^2, w - w^2 + x^2w^3)$. Undoing the two blow-up, the origin of the vector field X_{12} has the local phase portrait of figure 7(b).

Summarizing we have

Proposition 19. *The phase portrait of X_{12} is topologically equivalent to*

(X12.1) Figure 1$(X_{12.1})$ *if* $b \neq 0$.
(X12.2) Figure 1$(X_{12.2})$ *if* $b = 0$.

The vector field X_{13}. The vector field coincides with vector field $X_{11.2}$ interchanging the variables x and y and reversing the time.

Summarizing we have

Proposition 20. *The phase portrait of X_{13} is topologically equivalent to*

(X13) Figure 1$(X_{11.2})$.

The vector field X_{14}. X_{14} coincides with vector field $X_{12.2}$ changing the variables x into y and y into $-x$.

Summarizing we have

Proposition 21. *The phase portrait of X_{14} is topologically equivalent to*

(X14) Figure 1$(X_{12.2})$.

The vector field X_{15}. X_{15} has three singular points, namely $(-1, 0)$ is a hyperbolic unstable node; $(-1, 1)$ is a hyperbolic saddle; $(0, 0)$ is linearly zero. To study this singular point we do the blow-up $(x, y) \to (x, v)$, where $y = vx$, and the vector field X_{15} becomes $(x^2 + x^3, -xv - xv^2)$. Rescaling the independent variable by x we get the vector field $(x + x^2, -v - v^2)$, which on the straight line $x = 0$ has two singular points, namely $(0, 0)$, $(0, -1)$. The first is a hyperbolic saddle and the second is a hyperbolic unstable node. Undoing the blow-up, the origin of the vector field X_{15} has the local phase portrait of figure 6.

Summarizing, we have

Proposition 22. *The phase portrait of* X_{15} *is topologically equivalent to*

(X15) Figure 1(X_{15}).

The vector field X_{16}. X_{16} has only the origin as a singular point which is linearly zero. To study this singular point we do the blow-up $(x, y) \to (x, v)$, where $y = vx$, and the vector field X_{16} becomes $(x^3, -xv^2)$. Rescaling the independent variable by x we get the vector field $(x^2, -v^2)$, which is homogeneous. So, its phase portrait is one of the six described in figure 2. Since $xQ - yP = xv(x + v)$, the phase portrait has three invariant straight lines through the origin, namely $x = 0$, $y = 0$ and $x + v = 0$. Due to the fact that this vector field at U_1 has a hyperbolic stable node and a hyperbolic saddle, it follows that the phase portrait of this vector field is the one of figure 2(f). Undoing the blow-up, the origin of the vector field X_{16} has the local phase portrait of figure 7(a).

Summarizing, we have

Proposition 23. *The phase portrait of* X_{16} *is topologically equivalent to*

(X16) Figure1(X_{16}).

Acknowledgments

The second author is partially supported by a DGICYT grant number MTM2005-06098-C02-01 and by a CICYT grant number 2005SGR 00550.

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